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A POISSON CONVERGENCE THEOREM FOR A PARTICLE SYSTEM WITH DEPENDENT CONSTANT VELOCITIES

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Consider an infinite collection of particles travelling in d-dimensional Euclidean space and let X_n denote the initial position of the n^{th} particle. Assume that the n^{th} particle has through all time the random velocity V_n and that $\{V_n\}$ is a sequence of dependent random variables. Let $X_n(t) = X_n + V_n t$ denote the position of the n^{th} particle at time t. Conditions are obtained for the convergence of $\{X_n(t)\}$ to a Poisson process as $t \to \infty$. Essentially they require that the dependence in the V_n-sequence decrease with increasing distance between the initial positions and that the conditional distribution of V_n given the initial positions of all the particles and V_k, $k \neq n$ be absolutely continuous with respect to Lebesgue measure.

particle systems	Poisson process
dependent motions	traffic flow models
weak convergence	

1. Introduction

Consider an infinite system of particles travelling in d-dimensional Euclidean space $\mathbf{R}^d$ where d is a positive integer. Let X_n be the initial position of the n^{th} particle and V_n be its constant (but random) velocity; X_n and V_n are d-dimensional random vectors. Put $X_n(t) = X_n + V_n t$, the position of the n^{th} particle at time t. Let $\mu_t(B)$ be the number of particles in the set B at time t. We will call the random measure μ_t the configuration of the positions at time t.

Under a fairly wide set of circumstances, we find that in practice the configuration of such an infinite particle system with random constant velocities can be approximated quite well by a Poisson process. An explanation of this Poisson tendency has been offered by Breiman [1], Dobrushin [4], and Stone [9] who show that, under certain "uniformity" conditions on the initial configuration μ_0, if the velocities V_n are independent of each other and of μ_0, then μ_t approaches a Poisson random measure as $t \to \infty$.

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The assumption of independent velocities is unrealistic in many cases. Our object is to extend their results by showing that the same result is true even when the velocities V_n depend on each other, provided that the dependence decreases with increasing distance between the initial positions. In other words, if at time 0, the particles had to choose velocities which by necessity depended on the velocities of nearby particles, then under certain conditions we would still observe the Poisson tendency for the configuration of the positions of the particles. Our result can also be used to explain the occurrence of a Poisson process as a suitable model for a process of scheduled arrivals with random delays having a large dispersion in the case in which the delays between arrivals that are scheduled to be close to one another may be correlated

To illustrate the type of results we obtain, consider the configuration of vehicles on a highway. In this case we take X_n and V_n to be real valued random variables. Suppose that, at time 0, the headways $X_{n+1}-X_n$ are independent identically distributed random variables and that the drivers choose their velocities as follows. The n^{th} driver considers the headway $X_{n+1}-X_n$ in front of him; if $X_{n+1}-X_n \geqslant \delta$ for some fixed length δ, then the velocity V_n is "chosen from" a density function p independent of all else; otherwise, if $X_{n+1}-X_n < \delta$, then V_n is picked from a density $q(V_{n+1},\cdot)$ depending on the velocity V_{n+1} of the vehicle in front. Under certain conditions (see Example (4.1) for the precise statement) we show that the configuration μ_t at time t is approximately Poisson for large t. Of course, in reality, the drivers will change their speeds as soon as they no longer feel impeded and will pick velocities from the desired density p. Although this is at variance with our model, its effect is to increase the speed of convergence of μ_t to a Poisson random measure; and hence, our results describe the worst that can happen.

In the next section we give a precise formulation of the problem and state our main result. Section 3 contains its proof. In Section 4 we present some examples. In one of them the hypotheses of the main result are not satisfied and we have convergence to a clustered Poisson process.

2. Preliminaries and the main result

Let $(\Omega, \mathcal{M}, P)$ be a complete probability space. We will write $\mathbf{N}=\{0,1,2,\ldots\}$, $\mathbf{R}_+=[0,\infty)$, $\mathbf{R}^d=(-\infty,+\infty)^d$, $\mathbf{Z}=\{0,\pm 1,\pm 2,\ldots\}$, and $\mathbf{Z}^d$ will be the integer lattice in $\mathbf{R}^d$. We will denote the Borel subsets of $\mathbf{R}_+$, (respectively $\mathbf{R}^d$), by $\mathcal{R}_+$, (respectively $\mathcal{R}^d$). Further, we will let $\mathbf{B}=[0,1]^d$ be the unit cube in $\mathbf{R}^d$ and $\mathbf{B}_0=[-1,1]^d$ be the cube of side 2 centered at 0. Before we give the main result we will give some preliminary definitions and assumptions.

2.1. Poisson random measures

A random measure μ on $(\mathbf{R}^d, \mathcal{R}^d)$ is said to be a Poisson random measure (PM) with rate $\lambda > 0$ provided that $\mu(A_1),\ldots,\mu(A_n)$ are independent whenever the

Borel subsets $A_1, \ldots, A_n$ of $\mathbf{R}^d$ are disjoint, and that $\mu(A)$ has the Poisson distribution with parameter $\lambda|A|$ where $|A|$ denotes the Lebesgue measure of A.

2.2. *Position and velocity sequences*

Let (X_n) and (V_n) be two sequences of random variables taking values in $\mathbf{R}^d$, and let

$$\mathcal{F} = \sigma(X_n\,;\, n \in \mathbf{N}), \qquad \mathcal{G}_k = \mathcal{F} \vee \sigma(V_n\,;\, n \in \mathbf{N}\backslash\{k\}) \tag{2.1}$$

where $\sigma(\cdot)$ denotes the σ-algebra generated by $(\cdot)$.

Concerning the velocities V_n we suppose that, for any $A \in \mathcal{R}^d$

$$\mathbf{P}\{V_n \in A \mid \mathcal{F}\} = \pi(A) = \int_A p(v)\mathrm{d}v \tag{2.2}$$

and

$$\mathbf{P}\{V_n \in A \mid \mathcal{G}_n\} = Q_n(\cdot\,, A) = \int_A q_n(\cdot\,, v)\mathrm{d}v \tag{2.3}$$

for some distribution π with density p and some transition probability Q_n from $(\Omega, \mathcal{G}_n)$ into $(\mathbf{R}^d, \mathcal{R}^d)$ which has a density $q_n : \Omega \times \mathbf{R}^d \to [0, b]$ jointly measurable in (ω, v) and bounded by some fixed number $b > 0$. In other words, although the velocities (V_n) are not independent, the distribution of a single velocity is independent of the positions of all the particles. The boundedness of q_n implies that

$$\sup_n \mathbf{P}\{X_n(t) \in A\} \leqslant bt^{-d}|A| \tag{2.4}$$

thus putting an upper bound on the probability that a particle is in the set A at time t.

2.3. *The initial configuration*

We now introduce conditions on the initial configuration $\mu = \mu_0$. Recall that $\mathbf{B} = [0, 1]^d$ is the unit cube in $\mathbf{R}^d$ and that for any scalar $s \in \mathbf{R}$ and vector $x \in \mathbf{R}^d$ we write $x + s\mathbf{B}$ for the set of all points of the form $x + sy$ with $y \in \mathbf{B}$.

2.5. Conditions. (a) The random measure μ is stationary and ergodic;

(b) For any $a \in (0, 1)$,

$$\lim_{s\to\infty} \frac{\mathbf{E}[\mu(s^a\mathbf{B})^2]}{s^{(1+a)d}} = 0.$$

Condition (2.5b) limits the growth of the second moment of the number of particles in $s^a\mathbf{B}$ as a function of s. Condition (2.5a) implies that there exists a constant $\lambda > 0$ such that

$$\lim_{s\to\infty} \mathbf{E}\left[\left|\frac{\mu(x - s\mathbf{B})}{s^d} - \lambda\right|\right] = 0 \tag{2.6}$$

uniformly in $x \in \mathbf{R}^d$; Debes [3]. This is the L^1-convergence assumption on μ used by Stone [9].

2.4. The structure of dependence between velocities

For Borel sets $A \subset B$ in $\mathbf{R}^d$ put $|B^c - A| = \inf\{|x-y|: x \in A, y \in B^c\}$. Let W and Z be positive random variables bounded above by 1 such that Z is $\sigma(V_i I_A(X_i); i \in \mathbf{N})$-measurable and W is $\sigma(V_i I_{B^c}(X_i); i \in \mathbf{N})$-measurable; ($I_A(x) = 1$ if $x \in A$ and 0 if $x \notin A$). We now introduce a condition concerning the conditional independence of W and Z given $\mathscr{F}$

2.7. Condition. There is a nonincreasing nonnegative function ϕ on $\mathbf{R}_+$ such that for all sets A and B in $\mathscr{R}^d$ with $A \subset B$ and all random variables W and Z as above

$$\mathbf{E}[|\mathbf{E}[ZW \mid \mathscr{F}] - \mathbf{E}[Z \mid \mathscr{F}]\mathbf{E}[W \mid \mathscr{F}]|] \leq \phi(|B^c - A|)$$

and

$$\lim_{s \to \infty} s^d \phi(s^c) = 0 \tag{2.8}$$

for some $c \in (0,1)$.

Condition (2.7) is a type of ϕ-mixing condition for the conditional joint distribution of the velocities of particles whose initial positions are in the set A and the velocities of particles whose initial positions are in the set B^c given the positions of all the particles at time 0. Roughly, it states that if $|B^c - A|$ is large, then W and Z are conditionally independent given $\mathscr{F}$ almost surely. Condition (2.7) is satisfied, for example, if the velocity of a particle is independent of the velocities of particles more than a distance δ away.

The following is the main result of this paper.

2.9. Theorem. *If conditions* (2.2), (2.3), (2.5), *and* (2.7) *are satisfied, then*

$$\lim_{t \to \infty} \mathbf{P}\{\mu_t(A_i) = k_i;\ i = 1, \ldots, n\} = \mathbf{P}\{\nu(A_i) = k_i;\ i = 1, \ldots, n\}$$

for all finite sequences of bounded sets $\{A_i; i = 1, \ldots, n\}$ *in* $\mathscr{R}^d$ *with* $|\partial A_i| = 0$, $i = 1, \ldots, n$ *where* ν *is a PM with rate* λ *and* ∂A_i *denotes the boundary of the set* A_i.

We will give the proof of this result in the next section. In the last section we will present three examples: two in which the hypotheses of the Theorem are satisfied and one in which the hypothesis (2.3) is not satisfied. In the latter we have convergence to a clustered Poisson process rather than a Poisson process.

3. Proof of the main result

We will give the proof of Theorem 2.9 in a series of lemmas. Recalling that $\mathbf{B}_0 = [-1, 1]^d$, we will assume initially that there is a velocity $v_0 > 0$ such that

$$\pi(v_0\mathbf{B}_0) = 1; \tag{3.1}$$

that is, the velocities of the particles are bounded.

Fix a bounded set $A \in \mathcal{R}^d$. Then there is a constant $a > 0$ such that $A \subset a\mathbf{B}_0$. Let $\beta_t = t^c$, $\alpha_t = t^{c+(1-c)/2}$, and $\gamma_t = \alpha_t + \beta_t$ where c is the constant in condition (2.7). Put

$$M_t = \{m \in \mathbf{Z}^d : \{m\gamma_t + \gamma_t\mathbf{B}\} \cap (a + v_0 t)\mathbf{B}_0 \neq \emptyset\}. \tag{3.2}$$

M_t is the collection of vectors $m \in \mathbf{Z}^d$ such that the intersection of the shifted smaller hypercube $m\gamma_t + \gamma_t\mathbf{B}$ and the larger set $(a + v_0 t)\mathbf{B}_0$ is not empty. Note that

$$\operatorname{Card} M_t \leqslant 2^d \theta\{(a + v_0 t)\gamma_t^{-1}\}^d \tag{3.3}$$

where

$$\theta\{x\} = \inf\{n \in \mathbf{N} : n \geqslant x\} \quad \text{for } x \in \mathbf{R}_+.$$

Let Γ^t be the hypercube $z + \alpha_t\mathbf{B}$ where z is the vector in $\mathbf{R}^d$ each of whose components is $\frac{1}{2}(\gamma_t - \alpha_t)$; that is, Γ^t is the hypercube with sides of length α_t that is centered in the larger hypercube $\gamma_t\mathbf{B}$. For each $m \in \mathbf{Z}^d$ let $\Gamma^t_m = m\gamma_t + \Gamma^t$ and $\Lambda^t_m = \{m\gamma_t + \gamma_t\mathbf{B}\} - \Gamma^t_m$. Γ^t_m is the hypercube with sides of length α_t that is centered in the larger hypercube $m\gamma_t + \gamma_t\mathbf{B}$. Put

$$Z^t_m = \sum_k I_{\Gamma^t_m}(X_k) I_A \circ X_k(t)$$

and

$$W^t_m = \sum_k I_{\Lambda^t_m}(X_k) I_A \circ X_k(t);$$

that is, Z^t_m (respectively W^t_m), is the number of particles that were in Γ^t_m (respectively Λ^t_m), at time 0 and are in A at time t.

We will show that as $t \to \infty$ the random variables $\{Z^t_m;\ m \in M_t\}$ are asymptotically independent;

$$\lim_{t\to\infty} \sum_{m\in M_t} \mathbf{P}\{Z^t_m = 1\} = \lambda\,|A|; \qquad \lim_{t\to\infty} \sum_{m\in M_t} \mathbf{P}\{Z^t_m \geqslant 2\} = 0;$$

and the probability law of $\mu_t(A)$ nearly equals that of $\sum_{m\in M_t} Z^t_m$ for t sufficiently large.

3.4. Lemma. *If* (2.3) *and* (2.5) *hold, then*

$$\lim_{t\to\infty} \sum_{M_t} \mathbf{E}[Z^t_m;\ Z^t_m \geqslant 2] = 0 \tag{3.5}$$

and

$$\lim_{t\to\infty} \sum_{M_t} \mathbf{E}[W^t_m] = 0. \tag{3.6}$$

Proof. Note that

$$\mathbf{P}\{X_n(t) \in A \mid \mathcal{G}_n\}(\omega) = Q_n\left(\omega, \frac{A - X_n(\omega)}{t}\right) \leqslant b\,|A|\,t^{-d}$$

by (2.4). Hence, putting $\eta_t(A) = b|A|t^{-d}$

$$\mathbf{P}\{Z'_m = k \mid \mathcal{F}\} \leq \binom{\mu(\Gamma'_m)}{k}\eta_t(A)^k \quad \text{for } k \leq \mu(\Gamma'_m).$$

Therefore,

$$\mathbf{E}[Z'_m; Z'_m \geq 2] \leq \mathbf{E}[\mathrm{O}(\mu(\Gamma'_m)^2\eta_t(A)^2)].$$

(3.5) now follows from (3.3) and (2.5). Similarly,

$$\sum_{M_t} \mathbf{E}[W'_m] = \sum_{M_t} \mathbf{E}\left[\sum_k I_{\Lambda'_m}(X_k)Q_k\left(\cdot, \frac{A - X_k}{t}\right)\right]$$

$$\leq \eta_t(A)\sum_{M_t} \mathbf{E}[\mu(\Lambda'_m)] \sim K(\gamma_t^d - \alpha_t^d)\gamma_t^{-d}$$

for some constant K since μ is stationary. The result now follows from the definitions of γ_t and α_t.

3.7. Lemma. *If conditions* (2.2), (2.3), (2.5), *and* (3.1) *hold, then*

$$\lim_{t\to\infty} \mathbf{E}\left[\left|\sum_{M_t} \mathbf{P}\{Z'_m = 1 \mid \mathcal{F}\} - \lambda|A|\right|\right] = 0.$$

Proof.

$$\mathbf{E}\left[\left|\sum_{M_t} \mathbf{P}\{Z'_m = 1 \mid \mathcal{F}\} - \lambda|A|\right|\right] \leq \mathbf{E}\left[\left|\sum_{M_t} \mathbf{P}\{Z'_m = 1 \mid \mathcal{F}\} - \sum_k \pi\left(\frac{A - X_k}{t}\right)\right|\right]$$
$$+ \mathbf{E}\left[\left|\sum_k \pi\left(\frac{A - X_k}{t}\right) - \lambda|A|\right|\right].$$

The hypotheses of Theorem 5 of Stone [9] are satisfied by (2.2). Hence, the hypotheses of Theorem 2 of Stone [9] are satisfied by (2.2) and (2.6) and we have

$$\lim_{t\to\infty} \mathbf{E}\left[\left|\sum_k \pi\left(\frac{A - X_k}{t}\right) - \lambda|A|\right|\right] = 0.$$

By (2.2), the definition of M_t, and (3.1)

$$\sum_k \pi\left(\frac{A - X_k}{t}\right) = \sum_k \mathbf{P}\{X_k(t) \in A \mid \mathcal{F}\}$$

$$= \sum_{M_t} \mathbf{P}\{Z'_m = 1 \mid \mathcal{F}\} + \sum_{M_t} \mathbf{E}[Z'_m; Z'_m \geq 2 \mid \mathcal{F}] + \sum_{M_t} \mathbf{E}[W'_m \mid \mathcal{F}].$$

The result now follows from Lemma (3.4).

3.8. Proposition. *If the hypotheses of Lemma* 3.7 *are satisfied and* (2.7) *holds then for* $a > 0$

$$\lim_{t\to\infty} \mathbf{E}\{\exp\{-a\mu_t(A)\}\} = \exp\{-(1 - \mathrm{e}^{-a})\lambda|A|\}.$$

Proof. Let $Y^t = \Sigma_{M_t} W^t_m$. Fix $\varepsilon > 0$ and let $N^\varepsilon_t = \{Y^t > \varepsilon\}$. Then, by (3.1)

$$\lim_{t\to\infty} \mathbf{E}[\exp\{-a\mu_t(A)\}] = \lim_{t\to\infty} \mathbf{E}\left[\exp\left\{-a\sum_{M_t} Z^t_m - aY^t\right\}\right]$$

$$\geqslant \lim_{t\to\infty} \mathbf{E}\left[\exp\left\{-a\sum_{M_t} Z^t_m - aY^t\right\};\, N^\varepsilon_t\right]$$

$$+ \lim_{t\to\infty} \mathbf{E}\left[\exp\left\{-a\sum_{M_t} Z^t_m - a\varepsilon\right\};(N^\varepsilon_t)^c\right].$$

By (3.6) $\lim_{t\to\infty} \mathbf{P}(N^\varepsilon_t) = 0$. Hence

$$\lim_{t\to\infty} \mathbf{E}[\exp\{-a\mu_t(A)\}] \geqslant \lim_{t\to\infty} \mathbf{E}\left[\exp\left\{-a\sum_{M_t} Z^t_m - a\varepsilon\right\}\right].$$

Since ε is arbitrary,

$$\lim_{t\to\infty} \mathbf{E}[\exp\{-a\mu_t(A)\}] = \lim_{t\to\infty} \mathbf{E}\left[\exp\left\{-a\sum_{M_t} Z^t_m\right\}\right].$$

By (2.7),

$$\mathbf{E}\left[\left|\mathbf{E}\left[\exp\left\{-a\sum_{M_t} Z^t_m\right\} \middle| \mathcal{F}\right] - \prod_{M_t} \mathbf{E}[\exp\{-aZ^t_m\} \mid \mathcal{F}]\right|\right] \leqslant \operatorname{Card}(M_t)\phi(t^c)$$

which tends to 0 as $t \to \infty$ by (2.8).

Let

$$p^t_m = (1 - e^{-a})\mathbf{P}\{Z^t_m = 1 \mid \mathcal{F}\} + \sum_{k=2}^{\infty} (1 - e^{-ak})\mathbf{P}\{Z^t_m = k \mid \mathcal{F}\}.$$

Then

$$\prod_{M_t} \mathbf{E}[\exp\{-aZ^t_m\} \mid \mathcal{F}] = \prod_{M_t} (1 - p^t_m).$$

Note that

$$\mathbf{E}\left[\sum_{M_t} (p^t_m)^2\right] \leqslant \sum_{M_t} \mathbf{E}[O(\eta_t(A)^2 \mu(\Gamma^t_m)^2)]$$

which tends to zero as $t \to \infty$ by (2.5b).

Let $\Omega^t_\varepsilon = \{\Sigma_{M_t}(p^t_m)^2 < \varepsilon\}$. Then for ε sufficiently small

$$\mathbf{E}\left[\exp\left\{-\left(\sum_{M_t} p^t_m - \varepsilon\right)\right\};\, \Omega^t_\varepsilon\right] + \mathbf{E}\left[\prod_{M_t} (1 - p^t_m);\, (\Omega^t_\varepsilon)^c\right] \leqslant$$

$$\leqslant \mathbf{E}\left[\prod_{M_t} (1 - p^t_m)\right] \leqslant \mathbf{E}\left[\exp\left\{-\sum_{M_t} p^t_m\right\}\right].$$

Since ε is arbitrary,

$$\lim_{t\to\infty} \mathbf{E}\left[\prod_{M_t} (1 - p^t_m)\right] = \lim_{t\to\infty} \mathbf{E}\left[\exp\left\{-\sum_{M_t} p^t_m\right\}\right].$$

The result now follows from Lemmas (3.4) and (3.7).

Proof of Theorem 2.9. By the remark following Proposition 3.2 of Jagers [7], $\{\mu_t\}$ is tight if and only if $\{\mu_t(C)\}$ is tight for all compact $C \in \mathcal{R}^d$. Since the set $A \in \mathcal{R}^d$ of Proposition 3.8 was an arbitrary bounded set, $\{\mu_t\}$ is tight under the added hypothesis 3.1. The result of Theorem 2.9 under the added condition 3.1 now follows by a result of Renyi's [6] in the version of Proposition 4.2 of Jagers [7] from Proposition 3.8.

It remains to dispose of condition 3.1. For an arbitrary stationary sequence $\{V_n\}$ satisfying the hypotheses of the Theorem 2.9 and any $\varepsilon > 0$ pick v_0 so that

$$1 - \pi(v_0\mathbf{B}_0) < \varepsilon.$$

Define a new set of random variables V_k' by

$$V_k' = \begin{cases} V_k & \text{if } V_k \in v_0\mathbf{B}_0, \\ Y_k & \text{if } V_k \notin v_0\mathbf{B}_0. \end{cases}$$

where $\{Y_k\}$ are independent random variables each having a uniform distribution on $v_0\mathbf{B}_0$. Then $\{V_k';\ k \in \mathbf{N}\}$ satisfies the hypotheses of Theorems 2.9 and 3.1. Put $X_k'(t) = \dot{X}_k + V_k't$ and $\mu_t' = \Sigma_k\, \varepsilon_{X_k'(t)}$; ($\varepsilon_x(B) = 1$ if $x \in B$ and 0 otherwise for $B \in \mathcal{R}^d$). We have shown that μ_t' converges weakly to a PM with rate λ as $t \to \infty$.

Let A be a bounded set in $\mathcal{R}^d$. Put

$$D_k(A) = \{X_k(t) \in A,\ X_k'(t) \notin A\},$$

$$D_k'(A) = \{X_k'(t) \in A,\ X_k(t) \notin A\},$$

and $C = (v_0\mathbf{B}_0)^c$. Then

$$\mathbf{P}\{D_k(A)\} \leq \mathbf{P}\{X_k(t) \in A,\ V_k \in C\} = \mathbf{E}\left[\int_C I_{A-ut}(X_k)\pi(\mathrm{d}u)\right].$$

Hence,

$$\sum_k \mathbf{P}\{D_k(A)\} \leq \mathbf{E}\left[\int_C \mu(A - ut)\pi(\mathrm{d}u)\right] = K\,|A|\,\pi(C) \leq K\,|A|\,\varepsilon$$

for some $K > 0$ since μ is stationary. Similarly

$$\mathbf{P}\{D_k'(A)\} \leq \mathbf{P}\{X_k'(t) \in A,\ V_k \in C\} = \mathbf{P}\{X_k + Y_kt \in A,\ V_k \in C\}.$$

Hence,

$$\sum_k \mathbf{P}\{D_k'(A)\} \leq \frac{\varepsilon}{|v_0\mathbf{B}_0|}\int_{v_0\mathbf{B}_0} \mathbf{E}[\mu(A - ut)]\mathrm{d}u = K\,|A|\,\varepsilon.$$

Therefore, for all $t > 0$,

$$\mathbf{P}\{\mu_t(A) \neq \mu_t'(A)\} \leq 2\,|A|\,K\varepsilon$$

from which it follows that

$$|\mathbf{P}\{\mu_t(A) = j\} - \mathbf{P}\{\mu_t'(A) = j\}| \leq 2\,|A|\,K\varepsilon$$

for all $t > 0$ and $j \in \mathbf{N}$. Since ε is arbitrary we have

$$\lim_{t\to\infty} \mathbf{P}\{\mu_t(A) = j\} = e^{-\lambda|A|} \frac{\lambda|A|^j}{j!}.$$

The result now follows from the arguments in the first paragraph of the proof.

4. Examples

In this section we present some examples. The first is the application of Theorem (2.9) to the traffic model alluded to in the introduction. The third gives a case in which the assumptions of Theorem (2.9) are not satisfied and we have convergence to a Poisson cluster process rather than to a Poisson process. In the second example the assumptions of Theorem (2.9) are satisfied but the dependence of the velocities is not Markovian.

4.1. Example. Let (X_n) be a sequence of real-valued random variables such that

$$\cdots \leqslant X_{-2} \leqslant X_{-1} \leqslant 0 \leqslant X_1 \leqslant X_2 \leqslant \cdots.$$

X_n is interpreted as the position of the nth vehicle on a highway at time zero. Suppose that $\{X_{n+1} - X_n;\ n \geqslant 1\}$ and $\{X_{-n} - X_{-n-1};\ n \geqslant 1\}$ are independent sequences of independent random variables each with the same distribution. We assume that the random measure μ induced by the positions of the vehicles at time zero satisfies conditions (2.5).

Let V_n be the constant velocity of the n^{th} vehicle. We assume the velocities of all the vehicles are nonnegative and that the driver of the n^{th} vehicle chooses his constant velocity at time 0 in the following manner. There is a $\delta > 0$ so that $\mathbf{P}\{X_2 - X_1 > \delta\} > 0$; that is, there is a positive probability of the headway between two vehicles being greater than δ. For $n \geqslant 1$, $n \leqslant -2$, and $A \in \mathcal{R}$,

$$\mathbf{P}\{V_n \in A \mid \mathcal{G}_n\} = \begin{cases} \pi(A) = \int_A p(x)\mathrm{d}x & \text{if } X_{n+1} - X_n \geqslant \delta, \\ \\ Q(V_{n+1}, A) = \int_A q(V_{n+1}, y)\mathrm{d}y & \text{if } X_{n+1} - X_n < \delta \end{cases} \tag{4.2}$$

and

$$\mathbf{P}\{V_{-1} \in A \mid \mathcal{G}_{-1}\} = \begin{cases} \pi(A) & \text{if } X_1 - X_{-1} \geqslant \delta, \\ Q(V_1, A) & \text{if } X_1 - X_{-1} < \delta \end{cases} \tag{4.3}$$

where π is a distribution having density p and Q is a transition probability from $(\mathbf{R}_+, \mathcal{R}_+)$ into itself having the bounded density function q. Further we assume that π is the unique invariant measure for Q. In other words, if the headway between the n^{th} and $(n+1)^{\text{st}}$ vehicle at time 0 is greater than or equal to δ, the n^{th} driver chooses his velocity from the density p independent of the other vehicles. If the headway is less than δ, his velocity will depend on the velocity of the vehicle in front of him.

4.4. Theorem. *μ_t converges weakly to a PM with rate λ as $t\to\infty$.*

Proof. Since π is the unique invariant measure for Q, conditions (2.2) and (2.3) are satisfied by (4.2) and (4.3). The conditions of 2.5 are also assumed. It remains to show that condition (2.7) is satisfied. Let $[a,b]$ be an interval of $\mathbf{R}$. Put

$$\underline{\psi} = \operatorname{Max}\{\sup(|X_{i+1}-X_i| : X_{i+1}, X_i \in [a,b]), \\ \inf(X_k - a\colon X_k \in [a,b]), \inf(b - X_k\colon X_k \in [a,b])\}.$$

If $Z \in \sigma(V_i I_{(-\infty,a)}(X_i))$ and $W \in \sigma(V_i I_{(b,\infty)}(X_i))$ are positive random variables bounded above by 1, then

$$|\mathbf{E}[ZW \mid \mathscr{F}] - \mathbf{E}[Z \mid \mathscr{F}]\mathbf{E}[W \mid \mathscr{F}]| \leq I_{[0,\delta)} \circ \underline{\psi};$$

that is, the velocities of vehicles whose initial positions are in $(-\infty, a)$ are independent of the velocities of vehicles whose initial positions are in (b,∞) if there is a headway in the interval $[a,b]$ which is greater than δ. Since at most $\theta((b-a)/\delta)-1$ intervals of length δ can fit into $[a,b]$

$$E\,[I_{[0,d)} \circ \underline{\psi}\,] \leq O(\mathbf{P}\{X_{n+1}-X_n < \delta\}^{\theta((b-a)/\delta)-1})$$

recalling that $\theta(x) = \inf\{n \in \mathbf{N}\colon n > x\}$. But $\mathbf{P}\{X_{n+1}-X_n<\delta\}<1$ by assumption and therefore (2.7) is satisfied for any $c \in (0,1)$.

In the next example the velocities of the particles are dependent but the dependence is not Markovian.

4.5. Example. Fix $1/\lambda > 0$ and put $X_n = n(1/\lambda)$, $n = 0, \pm 1, \pm 2, \ldots$. Let (S_n) be a sequence of independent exponential random variables with parameter $\mu > 0$. Fix $0 < \beta < 1$ and $0 \leq \rho < 1$. Let (U_n) and (W_n) be independent sequences of independent variables taking the values 0 and 1 with $\mathbf{P}(U_n = 0) = \beta$ and $\mathbf{P}(W_n = 0) = \rho$. Put

$$V_n = \beta S_n + U_n A_{n-1} \tag{4.6}$$

and

$$A_n = \sum_{k=-\infty}^{n} \rho^{n-k} W_k S_k. \tag{4.7}$$

(V_n) is a EARMA(1,1) process (exponential mixed autoregressive-moving average process of order 1), in the sense of Jacobs and Lewis [6]. In [6] it was shown that (V_n) is a stationary sequence of dependent exponential random variables with parameter λ. The process (V_n) is not Markovian in general.

We interpret X_n as the position of the n^{th} particle at time 0 and V_n as its random constant velocity. Then $X_n(t) = X_n + V_n t$ is the position of the n^{th} particle at time t. For $A \in \mathscr{R}$, let $\mu_t(A) = \sum_k I_A \circ X_k(t)$, the number of particles whose positions at time t are in A. Then μ_t is the random counting measure induced by the positions of the particles at time t.

4.8. Theorem. *μ_t converges weakly to a PM with rate λ as $t \to \infty$.*

Proof. The random counting measure μ induced by the X_n satisfies (2.5). We need to show that the other assumptions of Theorem 2.9 are satisfied. Note that by (4.7) (A_n) is a discrete time Markov process. By (4.6) we have that for $C \in \mathscr{R}_+$

$$
\begin{aligned}
&\mathbf{P}(V_n \in C \mid V_k;\, k \neq n) = \\
&\quad = \mathbf{E}[\mathbf{P}(V_n \in C \mid V_k;\, k \neq n,\, A_j;\, j \neq n-1)] \\
&\quad = \mathbf{E}\Big[\mathbf{E}\Big[\beta \int_C \lambda\,\mathrm{e}^{-\lambda x/\beta}\mathrm{d}x \\
&\qquad\qquad + (1-\beta)\int_{C \cap [A_{n-1},\infty)} \lambda\,\mathrm{e}^{-\lambda(x-A_{n-1})/\beta}\mathrm{d}x \;\Big|\; A_{n-2}, A_n\Big] \;\Big|\; V_k;\, k \neq n\Big].
\end{aligned}
$$

Thus the conditional distribution of V_n given $V_k;\, k \neq n$ has a bounded density.

In [6], it was shown that if $B = f(V_n, V_{n+1}, \ldots, V_{n+m})$ and $C = g(V_{n+m+k}, V_{n+m+k+1}, \ldots)$ are two events, then

$$|P(B \cap C) - P(B)P(C)| \leq a\rho^{\frac{1}{2}k}$$

for k sufficiently large and some constant $a > 0$. Hence condition 2.7 is satisfied for any $0 < c < 1$. The result now follows from Theorem 2.9.

In the last example the conditions of Theorem 2.9 are not satisfied.

4.9. Example. Fix $0 < \rho < 1$ and a constant $\delta = p/\lambda > 0$. Let the position of the n^{th} vehicle on a highway at time 0 be $n\delta$, $n \in \mathbf{Z}$; that is, the initial positions of the vehicles are deterministic, the distance between two vehicles being δ. We will assume that the nonnegative constant velocity V_n of the n^{th} vehicle is chosen as follows: with probability p the vehicle chooses its velocity from a distribution π that is absolutely continuous with respect to Lebesgue measure independent of the other vehicles; with probability $(1-p)$ it takes on the velocity of the first vehicle to the right of it at time 0.

More precisely, let (U_n) be a sequence of independent nonnegative random variables with distribution π. Let (Z_n) be a sequence of independent random variables taking the values 0 and 1 independent of (U_n) with $\mathbf{P}\{Z_n = 1\} = p$. Put $N_0 = 0$, $N_k = \inf\{n > N_{k-1}: Z_n = 1\}$, and $N_{-k} = \sup\{n < N_{-k+1}: Z_n = 1\}$ for $k \geq 1$. Then

$$V_0 = U_0, \qquad V_n = U_{N_k}, \qquad N_{k-1} < n \leq N_k.$$

Let $L_k = N_k - N_{k-1} - 1$ be the number of vehicles in the platoon following the N_k^{th} vehicle. For $A \in \mathscr{R}$ and $B \subset \mathbf{N}$ put $\nu_t(A \times B) = \sum_k I_A \circ X_{N_k}(t) I_B \circ L_k$; that is, ν_t

is the random counting measure induced by the positions at time t of the leaders of the platoons and the number of vehicles in the platoon.

Let $A_1, \ldots, A_n$ be bounded sets in $\mathcal{R}$ and $B_1, \ldots, B_n$ be finite sets in $\mathcal{N}$, the collection of subsets of $\mathbf{N}$. Put $D = \bigcup_k A_k \times B_k$. The arguments in the proof of Theorem 2.9 can be used to show that the distribution of $\nu_t(D)$ converges to a Poisson distribution with parameter $m \times G(D)$ as $t \to \infty$ where $m \times G$ is the product measure on $(\mathbf{R} \times \mathbf{N}, \mathcal{R} \times \mathcal{N})$ of $m(\cdot) = \lambda |\cdot|$ and G with $G(k) = (1-p)^k p$, $k = 0, 1, \ldots$. Hence by the arguments used before, ν_t converges weakly to a PM ν with mean measure $m \times G$; that is, for any $B \in \bar{\mathcal{R}} \times \mathcal{N}$ with $m \times G(B) < \infty$, $\nu(B)$ has a Poisson distribution with parameter $m \times G(B)$; if $m \times G(B) = \infty$, then $\nu(B) = \infty$ with probability one.

Since (X_n) is deterministic and $V_n = U_{N_k}$ for all $N_{k-1} < n \leq N_k$,

$$\mu_t(A) = \sum_k \sum_{i=0}^{L_k} I_A(X_{N_k}(t) - id)$$

for any Borel set A of $\mathbf{R}$. Hence μ_t converges weakly to a cluster process, (cf. [2]), the primary points of which form a Poisson process with rate λ; each primary point serves as the origin for a terminating renewal process on $(-\infty, 0]$ with interarrival distribution F on $(-\infty, 0]$ with

$$F(x) = \begin{cases} 0 & \text{if } |x| < \delta, \\ (1-p) & \text{if } |x| \geq \delta. \end{cases}$$

References

[1] L. Breiman, The Poisson tendency in traffic distribution, Ann. Math. Statist. **34** (1963) 308–311.

[2] D.R. Cox and P.A.W. Lewis, The Statistical Analysis of Series of Events (Methuen, London, and Barnes and Noble, New York, 1966).

[3] H. Debes, J. Kerstan, A. Liemant and K. Matthes, Verallgemeinerungen eines Satzes von Dobrushin I, Math. Nachr. **47** (1970) 183–244.

[4] R.L. Dobrushin, On Poisson laws for distributions of particles in space (in Russian), Ukrain. Math. Z. **8** (1956) 127–134.

[5] J.L. Doob, Stochastic Processes (Wiley, New York, 1953).

[6] P.A. Jacobs and P.A.W. Lewis, A mixed autoregressive-moving average exponential sequence and point process (EARMA (1,1)), Adv. Appl. Prob. **9** (1977).

[7] P. Jagers, Aspects of random measures and point processes. In Advances in Probability and Related Topics 3, ed. by P. Ney and S.C. Port (Marcel Dekker, New York, 1974) pp. 179–239.

[8] A. Renyi, Remarks on the Poisson process, Studia Sci. Math. Hungarica **2** (1967) 119–123.

[9] C. Stone, On a theorem by Dobrushin, Ann. Math. Statist. **39** (1968) 1391–1401.